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Target Competencies

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Table of Contents

1 INTRODUCTION.....	4
2 COMPETENCY ONTOLOGY.....	5
2.1 INTRODUCTION.....	5
2.2 THEORETICAL BACKGROUND	6
2.2.1 PISA 2000 international.....	6
2.2.2 The German national educational standards.....	10
2.2.3 ActiveMath.....	11
2.2.4 Mathematical bridging course at university of Kassel.....	14
2.3 THE COMPETENCY CLUSTERS IN MATH-BRIDGE.....	15
2.4 TOPICS FOR FUTURE WORK.....	18
3 ONTOLOGY.....	19
3.1 MOTIVATION.....	19
3.1.1 The use of the domain ontology in Math-Bridge.....	19
3.1.2 What is an ontology?.....	19
3.1.3 Taxonomy of mathematical topics.....	20
3.1.4 Conceptual framework of ActiveMath.....	21
3.2 THE DOMAIN ONTOLOGY.....	22
3.2.1 Existing implicit ontologies in the ActiveMath-context.....	22
3.2.2 The domain-ontology for Kassel material.....	23
3.2.3 Transformation of the mathematical taxonomy for Mathematical Sciences Education into a domain ontology.....	24
3.2.4 Problems and future work.....	24
4 SUMMARY.....	25
5 LITERATURE.....	26
6 ANNEX.....	28
6.1 ACTIVE MATH: MAPPINGS OF THE PROPOSED COGNITIVE PROCESSES TO REVISED BLOOM AND PISA.....	28
6.2 DETAILED DESCRIPTION OF THE COMPETENCY LEVELS ACCORDING TO THE COMPETENCIES IN PISA.....	29
6.3 DOMAIN ONTOLOGY FOR THE KASSEL MATERIAL.....	31
6.3.1 Set.....	31
6.3.2 Power.....	31
6.3.3 Equation.....	32
6.3.4 Function.....	32
6.3.5 Sequence.....	33
6.3.6 Limit.....	33
6.3.7 Overview of the symbols.....	34
6.4 TAXONOMY.....	35
6.4.1 Numbers and Computation.....	35
6.4.2 Logic and Foundations	35
6.4.3 Algebra and Number Theory.....	36
6.4.4 Discrete Mathematics	36
6.4.5 Geometry and Topology	37
6.4.6 Calculus.....	37
6.4.7 Analysis.....	38
6.4.8 Statistics and Probability.....	38
6.4.9 Applied Mathematics	39

1 Introduction

The competency model and the domain ontology classify the mathematical content to be used in the Math-Bridge context as a whole and set the common ground for all developments within the project. Both are needed for the content enrichment, use and accessibility, evaluation and exploitation as well as for dissemination. The target competencies that are defined in this report consist of a competency model in two dimensions and a domain ontology as a third dimension: The competency model defines a competency-hierarchy for the covered mathematical domains and cognitive processes. The domain ontology defines the content hierarchy and is essential for the semantic content representation. Both are needed for all work packages of the project.

Since Math-Bridge is an international project, which aims at a wide international range of use and reuse, the competency model and the domain ontology should be embedded in and acknowledged by the international scientific community. Thus the theoretical basis for both must be well-grounded in the international discussion and should use the common terms and definitions.

Beyond an introduction and a summary, this paper is structured in two main parts: one for the competency model and one for the domain ontology.

Concerning the competency model, we first discuss the most common international competency models, which provide a basis for the Math-Bridge competency model and which are necessary for a complete classification as well as for an easy and understandable user-feedback (subsection "2.2 Theoretical background"). Subsequently we describe Math-Bridge's competency model as a result from this theoretical review and from the Math-Bridge discussion process in subsection 2.3 "The competency ontology in Math-Bridge".

For defining the domain ontology, we first describe the motivation for a domain ontology. To this end we discuss the importance of the domain ontology for the upcoming work packages, the characteristics of ontologies in general and some examples and the conceptual framework given by ActiveMath (subsection "3.1 Motivation"). In the second step we present our conceptual decision based on the framework and present results of our work.

2 Competency ontology

2.1 Introduction

The competency ontology is the theoretical basis for the classification of mathematical competencies. It is needed for the metadata enrichment of the content, for defining the target competencies of the Math-Bridge users and finally for user feedback. Especially with regard to the last point, this classification is not a simple job to do.

All pedagogical partners of WP1 discussed a first version of the competency ontology at the launch workshop in Saarbrücken in May 2009. The first agreement on the competency ontology took into account the existing competency models from PISA international, the PISA related categorization for the German national standards for secondary mathematics education (KMK 2003).

The result of this first discussion was the agreement on a 3-dimensional competency ontology with the dimensions mathematical competencies, levels and mathematical domains:

1st dimension: mathematical competencies:

- represent
- solve mathematical problems
- communicate
- model real world problems
- reasoning

2nd dimension: competency levels:

- reproduce
- relate concepts
- reflect and generalise

3rd dimension: mathematical domains:

The details for the 3rd dimension have to be taken from the domain ontology.

During the launch workshop in Saarbrücken in May 2009 it was agreed that the partners from Kassel and Paderborn should explore the international discussion on competency models in order to substantiate the model that will be used in Math-Bridge.

Moreover we had to consider that the classification to be developed needs to be usable for the learner model as well as for the user feedback.

Hence under this focus, we studied again the competency models of PISA international, the categorization of the German national educational standards and in detail the competency model as well as the learner model already used in ActiveMath. All models are described briefly in the next section.

2.2 Theoretical background

2.2.1 PISA 2000 international

The competence model of PISA has three dimensions:

1st dimension: mathematical competencies

2nd dimension: competency clusters

3rd dimension: overarching ideas

According to Prenzel et al. (2003) and OECD (2003 & 2005) these three dimensions can be described as follows.

In the first dimension (mathematical competencies), PISA differentiates competencies that are needed to work on mathematical exercises. They have come to eight mathematical competencies, which rely on a classification by Niss (1999, pp. 21 - 29):

1st dimension: mathematical competencies

1. Thinking and reasoning
2. Argumentation
3. Communication
4. Modelling
5. Problem posing and solving
6. Representation
7. Using symbolic, formal and technical language and operations
8. Use of aids and tools

Since PISA hasn't focused on testing these competencies separately and since typical literacy-exercises usually need several of these competencies, the consortium restructured these cognitive activities and build three different **competency clusters** on a second dimension, in which the level of cognitive activities was described. These competency clusters constitute the second dimension of PISA's classification and are built upon each other.

2nd dimension: competency clusters

1. Reproduction cluster

“The competencies in this cluster essentially involve reproduction of practised knowledge. They include those most commonly used on standardised assessments and classroom tests. These competencies are knowledge of facts and of common problem representations, recognition of equivalents, recollection of familiar mathematical objects and properties, performance of routine procedures, application of standard algorithms and technical skills, manipulation of expressions containing symbols and formulae in standard form, and carrying out computations.” (OECD 2003, p. 42)

2. Connections

“The connections cluster competencies build on the reproduction cluster competencies in taking problem solving to situations that are not simply routine, but still involve familiar, or quasi-familiar, settings.” (OECD 2003, p. 43)

3. Reflexions

“The competencies in this cluster include an element of reflectiveness on the part of the student about the process needed or used to solve a problem. They relate to students' abilities to plan solution strategies and implement them in problem settings that contain more elements and may be more “original” (or unfamiliar) than those in the connections cluster.” (OECD 2003, p. 46)

For each of the 8 competencies, we find exemplary tasks and characterizations that specify what these 8 competencies mean on the 3 levels “reproduction”, “connections” and “reflexions”. OECD (2003, pp. 41 - 49). For every item, PISA internally decides which of the 8 competencies are involved in this item. For all these competencies an assignment to a competence cluster is made, which can be different for the 8 competencies. However, PISA summarizes this and only assigns one global competence cluster for each item. The differentiated assignments are predominantly used to interpret the empirical competency levels in a meaningful way (see below).

On the third dimension, PISA distinguishes the mathematical domain as follows:

Dimension 3: overarching ideas

1. quantity
2. space and shape
3. change and relationships
4. uncertainty

These four ideas are not identical to the curricular content strands that are usually described (e.g. algebra, geometry, etc.). They have been developed with regard to a phenomenological organisation of mathematics and aim at the detection of student's abilities to solve real world problems, the main goal of PISA.

These assignments are a priori classifications of the tasks. It is expected that the empirical difficulty of a tasks in the PISA population is somewhat correlated to the three competency clusters, however, an overlap was expected and empirically found.

PISA’s a posteriori “competency levels”

After the data had been gathered in the PISA project, the group assigned so-called competency levels to each item, which are often confounded with the a priori competency clusters. This is done in the following way.

Within the PISA testing, a one-dimensional score for each test student is calculated using the Rasch-model. The continuous scale is divided into six discrete **competency levels** by means that are described in detail in the PISA literature. Using the a priori descriptions of the competencies and competency cluster for the exercises, the competency levels can be interpreted in a subject matter related way.

Since for each item a score is calculated too, the items can be related to the same scale as the test students. The score of an exercise in PISA means, that a pupil with the same score will be able to solve the item correctly with a probability of about 68%.

PISA gives a feedback to the students concerning their abilities and knowledge, in the shape of the test score. This test score means among other that the student can solve such and such items with such and such a probability. According to this model a student with a score s can solve items with the same score s with a probability of about 65%.

In order to make these scores more intelligible the PISA authors have decided to reduce the data and put each student in one of the six competency levels. The description of the competency levels, which are quoted below can thus be interpreted as descriptions of the mathematical competence the student has. We can interpret this as PISA's student feedback.

We note that PISA does not communicate back to the students a competence profile according to the 8 competences or the mathematical domains but just one number, associated with a competence level. Moreover PISA does not claim that its tests provide reliable individual diagnostics but PISA is predominantly aiming at group comparisons.

The result of this interpretative description of the competency levels can be found in OECD (2005):

Overall mathematical literacy

- 6 At Level 6 students can conceptualise, generalise, and utilise information based on their investigations and modelling of complex problem situations. They can link different information sources and representations and flexibly translate among them. Students at this level are capable of advanced mathematical thinking and reasoning. These students can apply their insight and understandings along with a mastery of symbolic and formal mathematical operations and relationships to develop new approaches and strategies for attacking novel situations. Students at this level can formulate and precisely communicate their actions and reflections regarding their findings, interpretations, arguments, and the appropriateness of these to the original situations.
- 5 At Level 5 students can develop and work with models for complex situations, identifying constraints and specifying assumptions. They can select, compare, and evaluate appropriate problem-solving strategies for dealing with complex problems related to these models. Students at this level can work strategically using broad, well-developed thinking and reasoning skills, appropriate linked representations, symbolic and formal characterisations, and insight pertaining to these situations. They can reflect on their actions and formulate and communicate their interpretations and reasoning.
- 4 At Level 4 students can work effectively with explicit models for complex concrete situations that may involve constraints or call for making assumptions. They can select and integrate different representations, including symbolic, linking them directly to aspects of real-world situations. Students at this level can utilise well-developed skills and reason flexibly, with some insight, in these contexts. They can construct and communicate explanations and arguments based on their interpretations, arguments and actions.
- 3 At Level 3 students can execute clearly described procedures, including those that require sequential decisions. They can select and apply simple problem-solving strategies. Students at this level can interpret and use representations based on different information sources and reason directly from them. They can develop short communications reporting their interpretations, results and reasoning.
- 2 At Level 2 students can interpret and recognise situations in contexts that require no more than direct inference. They can extract relevant information from a single source and make use of a single representational mode. Students at this level can employ basic algorithms, formulae, procedures, or conventions. They are capable of direct reasoning and making literal interpretations of the results.
- 1 At Level 1 students can answer questions involving familiar contexts where all relevant information is present and the questions are clearly defined. They are able to identify information and to carry out routine procedures according to direct instructions in explicit situations. They can perform actions that are obvious and follow immediately from the given stimuli.

fig.: OECD 2005, p. 261.

In OECD (2005) these levels are interpreted with regard to the eight competencies in much more detail, see in Annex.

2.2.2 The German national educational standards

The German national educational standards, that were developed by the KMK (2003), are very similar to the competency model of PISA, but few differences can be found in the details.

On the first dimension, instead of the eight competencies in PISA, the KMK-model distinguishes the following 6 mathematical competencies:

1st dimension:

General mathematical competencies:

1. Mathematical argumentation
2. Mathematical problem solving
3. Mathematical modelling
4. Using mathematical representations
5. Dealing technically/ formally/ symbolically with mathematics
6. Mathematical communication

The two competencies "mathematical thinking" and "use of aids and tools", that can be found in the PISA classification are obviously missing in this list. According to a conversation with Prof. Dr. W. Blum, a member of the German PISA consortium and the international PISA board, these two categories didn't play a major role in PISA international as well. No exercises could be found that need these competencies specifically: "mathematical thinking" is hard to test and is implicitly part of other competencies and "use of aids and tools" is part of "dealing technically/ formally/ symbolically with mathematics".

In analogy to the three competency clusters in PISA, the educational standards from the KMK describes in a second dimension three **achievement levels** (in German "Anforderungsbereiche"):

Dimension 2: Achievement/cognitive complexity levels (KMK 2003, pp. 13):

(AI) Reproduce

- reproduction and direct application of basic ideas, rules, procedures in limited fields and in repetitive contexts

(AII) Establish relations

- work in known contexts,
- connection of knowledge, skills and capabilities that are known from mathematical contexts in different fields

(AIII) Generalise and reflect

- work on complex situations et al. with the goal to pose, solve, reason, conclude, interpret or to value a problem

In the third dimension, the mathematical domains are classified in five **overarching ideas** (in German: "**Leitideen**"), which are:

3rd dimension: overarching ideas

1. Numbers ("Zahl")
2. Measuring ("Messen")
3. Space and shape ("Raum und Form")
4. Functional relation ("funktionaler Zusammenhang")
5. Data and chance ("Daten und Zufall")

Different from PISA, the additional overarching idea "measuring" is introduced.

The classification of the exercises is done before testing. For each exercise the overarching idea and all mathematical competencies that are needed are identified. The national standards assign a global achievement level to each item. This global score is partly based on a more detailed analysis with regard to which competencies are required on which level for solving the item. This is a procedure similar to PISA.

It is planned to empirically investigate the achievements of the national standards in the German student population. This may lead to an additional classification of items by their empirical difficulty.

2.2.3 ActiveMath

The competency model of "ActiveMath" was originally based on the competency model by Bloom and ideas from Anderson et al. (2001) and Flavell (1979). Later in the project, the pedagogical partners introduced elements of the PISA categorization.

This had led to the following actual three-dimensional model according to Faulhaber et al. (2008).

2.2.3.1 Actual model of ActiveMath

1st dimension: taxonomy of cognitive processes

2nd dimension: competency levels

3rd dimension: domain ontology

1st dimension: taxonomy of cognitive processes

The cognitive processes that are distinguished in ActiveMath go back to the taxonomy by Bloom. Additionally concepts by Anderson et al. (2001) and ideas from Flavell (1979) were taken into account. Details of the ActiveMath classification can be found in the first and second column of table 1 in the Annex, which is directly taken from Faulhaber et al. (2008).

More detailed, ActiveMath distinguishes the following cognitive processes:

- Remember (basic retrieval operations: recognition of knowledge, recall from memory)

- Represent (interpret, illustrate and summarize (generalize) knowledge)
- Solve (estimate a result without calculating exact value, apply algorithms, apply tools)
- Analyse (differentiate between important and unimportant, organize information, attribute a bias, value or intent)
- Model (decode information, transform it into a mathematical model, encode a mathematical model into a situational context, generate hypotheses, produce new models by combining hypotheses)
- Communicate (how to describe one's own knowledge, argue about different aspects of knowledge, prove certain facts)
- Meta-cognition (reflect upon one's own knowledge and thinking processes, help seeking, search for information to fill knowledge gaps, detect errors, plan tasks by dividing into steps and order them, self-monitor actions and behaviour, self-explain)

2nd dimension: competency level

With regard to the second dimension, ActiveMath adopts competency levels from PISA (cf. Goguadze, 2004, p. 22):

The competency levels describe different levels of mathematical knowledge and abilities. ActiveMath applies these levels on examples and exercises as well as on different exercise steps.

The following **competency levels** are distinguished (cf. Goguadze, 2004, p. 22):

1. elementary
2. simple conceptual
3. multistep
4. complex

We see an analogy here to the 3 competency clusters in PISA or the achievement/cognitive complexity levels of the German national standards. In ActiveMath these competency levels are assigned to an item/task a priori and globally, i.e. no distinction of this level with regard to the above competencies is made.

The competencies and the level are assigned to all ActiveMath exercises or sub-exercises. This is important meta-data that is used to update the learner model, when a student has solved or not solved such an exercise.

Moreover, in ActiveMath the following 5 **difficulty levels** are assigned to items:

- very easy
- easy
- medium
- hard
- very hard

They describe, "*how hard it is to work through the learning object for the typical target audience*" (Gogvadze 2004, p. 19). Within the learner model, these difficulties are combined with the competency levels, so that each competency level subdivides into five difficulty levels. Thereby the competency levels overlap, so that an exercise on elementary level, that is rated as "very hard" can also be used as an "easy" exercise on the second level "simple conceptual".

These assignments are in analogy to PISA's competency 5 or 6 competency levels. The PISA score measures the empirical difficulty of an item in a certain target population.

In contrast to PISA, for the time being these difficulty levels in ActiveMath are assigned a priori to items, that means the competency level as well as the difficulty level of an exercise are not derived from empirical test results but from theoretical considerations.

The procedure for assigning difficulty levels, competency levels and cognitive processes to individual items has been done in ActiveMath by pedagogical experts, but without any detailed documentation.

3rd dimension: domain ontology:

With respect to the third dimension, the mathematical domain is described by the specific domain ontologies. For more detailed domain ontologies see also chapter 3 of this report.

2.2.3.2 Theoretical Background of the ActiveMath model

According to Faulhaber et al. (2008, pp. 2), the taxonomy developed by Bloom represents a hierarchy of the following six "educational goals":

- knowledge
- comprehension
- application
- analysis
- synthesis
- evaluation

Anderson et al. (2001) extended this taxonomy by introducing domain concepts as a second dimension.

In analogy to this approach and in analogy to the PISA and the German national standards approach, the Active Math group decided to conceptualize a competence as a pair consisting of cognitive processes and content/knowledge elements. This approach also reflected the following observation with respect to the description of learning goals in curricula: there the verb of a sentence defines a cognitive process and the nouns point to the domain ontology.

For describing the cognitive processes, ActiveMath uses the classification by Anderson et al. (2001), which was enriched by the meta-cognitive processes view of Flavell (1979, pp. 906 - 911). A reason for this was that

“the absence of meta-cognitive processes [...] does not allow for a differentiation of how well a student is able to consciously apply meta-cognitive operations to specific knowledge elements or within a certain domain, e.g. questions such as “how proficient is the student in detecting errors within problems of the fraction domain?”, cannot be answered.”

(Faulhaber et al. 2008, p. 5)

Faulhaber et al. pointed out, that meta-cognitive elements are not global but usually occur in a knowledge context. That's why they include the meta-cognitive processes in the process dimension, which is afterwards paired with knowledge elements.

The third basis of the ActiveMath's competency model is the categorization of PISA as described in a section above. Though the competence model of PISA was adopted, the incorporation raised up some problems, since there is no simple mapping between the cognitive processes listed above and the competencies defined in PISA.

“One reason for the difficulty is that the PISA-competencies often include multiple processes in a single competency and sometimes also elements that we would place into the knowledge dimension. Thus, for the mapping, we compared the aspects of PISA competencies regarding the actual cognitive processes involved. Elements that are placed into parentheses capture only part of the scope of a proposed cognitive process.”

(Faulhaber et al. 2008, p. 8)

The result of this mapping can be found in the table in the Annex.

2.2.4 Mathematical bridging course at university of Kassel

Since we see the competency ontology to be developed also as a basis for the feedback on the learners abilities, we describe in the following briefly the categorization used for the diagnostic testing and learner feedback in the project "Multimedivorkurs Mathematik".

The project "Multimedivorkurs Mathematik"/ "VEMA" was originated in March 2003 at the University of Kassel. Its objective is to supplement preparatory courses in mathematics that aim at facilitating the transition from school (or job) to university courses with mathematical content. Creating a computer-based self-learning facility allows the students to understand the content at an individualized learning speed (cf. Biehler/Fischer 2006).

Since 2007, a new e-learning course concept was designed and investigated, in which the students learn autonomously with the interactive learning material. For a sensitive support of self-learning, the user needs an individual feedback on his abilities and knowledge as well as a concrete hint for the next steps in his learning. Therefore we designed diagnostic tests that are automatically corrected, provide an immediate and differentiated feedback on the learners' abilities and knowledge and tell him/her what to learn next.

Since first-year students usually are no teachers or pedagogical experts, we aimed at developing an easy feedback that bases on the following two-dimensional feedback:

1st dimension: mathematical competence clusters

In the first dimension, we differentiated the following four **mathematical competence clusters**:

- **technical competence**
includes computing, solving of equations, simplification of terms or drawing the graph

of a function

- **understanding**
includes the understanding of the contents, to identify and to describe connections between concepts, to solve complex mathematical problems
- **application and modelling**
includes solving exercises with application of mathematics in the context of real world problems or in mathematical contexts
- **error diagnosis**
includes the identification of errors in mathematical solutions and argumentations. The learner has to analyse the mistakes in order to prevent them in the future, to reflectively understand the correct procedures, and to enhance his/her diagnostic competence (especially important for future teachers and tutors)

2nd dimension: mathematical domains (modules)

On the second dimension, we distinguish the content knowledge according to the modules that are the structural elements of our interactive book.

The items in the diagnostic tests in the Kassel material are classified according to these two dimensions. User-feedback consists of providing a score for all four competencies related to a mathematical domain/module associated with recommendations for further studies.

2.3 The competency clusters in Math-Bridge

In view of the categorization systems described above, we are suggesting a competency ontology that is both: a classification of mathematical content and competencies as a whole and a categorization that can be used for an understandable and helpful user-feedback.

Therefore we elaborated and modified the 3 dimensional model that was agreed on at the launch workshop in Saarbrücken.

In the first dimension, we describe mathematical competencies. We reduced the competency dimension from 5 to just 4 competencies as described below. This reduced model has the advantage of an easy classification and gives the possibility of an understandable feedback concerning the user's abilities. This will be an important point in concerning the usability of Math-Bridge. From our point of view, a more detailed classification is not needed for Math-Bridge and makes therefore no sense.

1st dimension: mathematical competence clusters

- 1. Technical competency**
(technical mastery of mathematical concepts and procedures)
- 2. Mathematical problem solving competency**
(mathematical problem solving and understanding)
- 3. Modelling competency**
(modelling of real world problems)
- 4. Communication and reasoning competency**
(communication, representation and explanation of mathematics)

For a better understanding, these competencies can be described as follows:

- 1. Technical competency**
 - This competency characterizes the technical mastery of mathematical concepts and procedures.
 - This competency requires for instance **technical** abilities such as:
 - drawing a graph
 - solving of equations
 - simplifying mathematical terms
 - use of computational (CAS, DGS, ...) and non-computational tools
 - differentiating a function
- 2. Mathematical problem solving competency**
 - This competency requires the understanding and solving of mathematical problems.
 - This competency requires
 - understanding of mathematical problems and concepts
 - solving mathematical problems
 - make conclusions on mathematical concepts
- 3. Modelling competency**
 - This competency relates to the modelling of real world problems.
 - This requires the following mathematical activities:
 - analysis of real world problems
 - transformation of the given situation to a reduced real model
 - mathematization of the reduced model to a mathematical model
 - interpretation of results in view of the real world situation
- 4. Communication and reasoning competency**
 - This competency reflects abilities that concern the communication, representation

and explanation of mathematical content.

- This competency requires
 - understanding and interpretation of mathematical concepts (definition, theorems, correlations of concepts, ...) and the ability to explain them
 - understanding and communicating mathematical proofs
 - communicating mathematical solving and results
 - understanding and application of mathematical reasoning

In comparison to the model the pedagogical partners agreed on in Saarbrücken, there are the following differences:

First, the competencies "represent", "communicate" and "reasoning" are integrated into the "**communication competency**" and are partly integrated into the "**mathematical problem competency**". Second, the new category "**technical competency**" is introduced and is differentiated from the category "**mathematical problem competency**". The introduction of the new category "technical competency" relies on our experiences in courses at our universities, especially in context of engineering. Within the lecturers it is consensus, that this competency is most important in particular in bridging courses and for the studies in all programmes. This competence relies on understanding concepts and may involve non-routine items. In contrast, however, the mathematical problem competency relates to items that directly address understanding and problem solving.

According to the decisions made in the meeting, we distinguish in **achievement/complexity levels** in the second dimension. They are derived from the competency clusters in PISA and the achievement levels in the German national educational standards:

2nd dimension: achievement levels

1. reproduction

- all competencies are on a reproductive level;
- all procedures are routine activities (computing, standard algorithms, technical skills,...)
- reproduction of known facts and procedures
- the problem is given in a common problem representation
- familiar mathematical contexts

2. connection

- builds on reproduction cluster
- at least one of the competencies is needed on an average level;
- it defines manageable activities with several steps
- problems based on no simple and routine situations
- quasi familiar settings

3. reflection

- builds on connection cluster

- at least one competency is needed on a high level;
- activities are complex;
- reflection and networking is needed
- planning and implementing is needed
- problem setting is close to real problems

In the third dimension, we describe the mathematical domains.

3rd dimension: domain-ontology

The elements of the 3rd dimension have to be taken from the domain ontology.

Beyond these three dimensions, **exercises** are to be classified in 5 difficulty levels that were also used in ActiveMath:

- very easy
- easy
- medium
- difficult
- very difficult.

2.4 Topics for future work

1. The next step in the project is to develop a codebook for the Math-Bridge competency model, which assures a fast, correct and standardised classification of the content within the project. Such a codebook should give exemplary examples and characterisations that will allow a reliable assignment of meta-data to all learning objects according to the suggested ontology.
2. An open question is whether every learning object gets assigned one or more competencies and whether we assign one global achievement level for every item or we assign achievement levels related to each competency.
3. From a pedagogical point of view, our model seems to be reasonable and helpful. Still it is necessary to check, whether the model is suitable to the metadata classification within the learner model of ActiveMath, too, and which adaptations are easily possible in ActiveMath. This has to be discussed in relation to the user model that will be implemented in the Math-Bridge system.
4. Generally this ontology system is related to the project a first draft that has to be adjusted later in the project.

3 Ontology

In this chapter we will describe the domain ontology used in Math-Bridge. To this end we describe first the motivation, the conceptional framework, the usage of the ontology and also an overview about existing ontologies. In a second subsection we present the result of our work. Moreover we figure out a few problems arising in the context of the domain ontology and in mathematical courses generally.

3.1 Motivation

In this section we describe our motivation for creating the domain ontology. In particular we comment on the usage of the ontology in our project and on the state of the art in building domain ontologies in mathematical contexts. Finally we describe the framework for the domain ontology given by ActiveMath.

3.1.1 The use of the domain ontology in Math-Bridge

First of all we have to discuss the question “why do we develop a domain ontology?” or in other words “What is the surplus for the project to develop a domain ontology?”

There are several items we have to invoke here:

1. The domain ontology structures the content for Math-Bridge. We structure the topics chosen by the content selection process. The structuring considers two aspects: first we define symbols that will be used in Math-Bridge, second we define is_a-Relations between these symbols. Within this step we define also names for the symbols.
2. By the domain ontology the process of slicing and the enrichment of the content with metadata will be more efficient. On this basis, it will be easier to assign the learning objects to several symbols and according to that to determine their object type, i.e. example, exercise, definition.
3. The domain ontology will be used to control the subject coverage in Math-Bridge. We will use the ontology for an overview about the types and the number of learning objects related to each symbol that are available in Math-Bridge.
4. A public browseable ontology can be used to give insight on the topics which are covered by MathBridge. It can also be used for navigation and to support searching.
5. The ontology will be enriched with individuals during content conversion phase.

3.1.2 What is an ontology?

In this section we will discuss the definition of an ontology:

“In the context of computer and information sciences, an ontology defines a set of representational primitives with which to model a domain of knowledge or discourse. The representational primitives are typically classes (or sets), attributes (or properties), and relationships (or relations among class members). The definitions of the representational primitives include information about their meaning and constraints on their logically consistent application.[...]”

“Due to their independence from lower level data models, ontologies are used for integrating heterogeneous databases, enabling interoperability among disparate systems, and specifying interfaces to independent, knowledge-based services.” (Gruber 2008)

Math-Bridge wants to develop an eLearning-System for remedial courses in Europe. Thus our domain is directed to the content of such remedial courses. For fixing this content, we started a process of content selection, which has already lead to a first version.

We think, it is a very important for defining an ontology to have existing “databases”. In our project the content partners bring in several “databases” with learning objects. We have to define our ontology with a view towards these databases and the required content for the remedial courses.

In the beginning we took a look at existing ontologies, just to learn and to work efficiently. So we searched for ontologies having something in common with the ontology to be developed. First we mention an ontology example, which is in so far related to our project as it provides an ontology for documents.

The Dublin Core project defines an ontology concerning cross-domain informations of a document like author, title, date, type, coverage ... In ActiveMath this ontology is used as a foundation for the metadata-definitions of our learning objects customized to our requirements. (cf. Gogvadze, 2004)

There are different domain ontologies, for instance for music (<http://musicontology.com/>), wine (<http://protege.cim3.net/file/pub/ontologies/wine/wine.owl>) and for biological subjects like genes, plants and proteins.

For mathematical contexts we find some ontologies about the representation of mathematical formulae like EngMath (<http://www-ksl.stanford.edu/knowledge-sharing/papers/engmath.html>), an ontology for mathematical models in engineering contexts.

Another domain ontology in mathematical contexts is the GeoSkills Ontology that has been developed in the InterGeo project (<http://i2geo.net/xwiki/bin/view/About/GeoSkills>). This domain ontology is used to categorize applets from dynamical geometric software.

3.1.3 Taxonomy of mathematical topics

With respect to mathematics there is a taxonomy for Mathematical Sciences Education (<http://people.uncw.edu/hermanr/mathtax/>). The version from May 2005 distinguishes between eleven categories and a number of subcategories within mathematics. The main categories are

- Numbers and Computation
- Logic and Foundations
- Algebra and Number Theory
- Discrete Mathematics
- Geometry and Topology
- Calculus
- Analysis
- Differential and Difference Equations
- Statistics and Probability
- Applied Mathematics
- Mathematics History

These categories are divided into several subcategories. In total, the taxonomy distinguishes nearly 400 topics.

This taxonomy is used for classifying digital resources in mathematics education.

In the launch workshop in Saarbrücken we agreed to use this taxonomy as a starting point for developing the Math-Bridge ontology. In a first step we have already used this taxonomy in our content selection process. Every content partner rated each topic by its importance for their study programmes (not important (0), medium important (1), very important (2)). After this process, we categorised the programmes into four categories (“engineering”, “mathematics and science”, “teachers” and “rest”) and calculated the mean of the ratings. By this method we ordered the topics by importance and selected those topics with a mean greater than 0.66 for Math-Bridge.

In a second step every content partner counted their learning objects for each topic. So we can now match the importance of a topic with the number of existing learning objects for this topic. This is important for estimating the effort for achieving a satisfactory coverage by learning objects for each topic.

This content selection provides the basis for the ontology to be developed.

3.1.4 Conceptual framework of ActiveMath

In this section we will discuss the conceptual frameworks we have to consider for our development. These frameworks are mainly conditioned by the use of ActiveMath and its characteristics. First we will describe the conditions set by ActiveMath, then the consequences for our ontology.

ActiveMath has a certain ontology of instructional objects, which defines a structure of learning objects (cf. Goguadze, 2004, p.6):

Symbols are the main structuring items. They are the major concepts the learner has to learn. For instance in Math-Bridge the users have to know the concept *linear function*. But this concept has to be characterised further, for instance it needs a definition, a motivation and some exercises and examples, none of them is a symbol. Consequently further general “types” of learning objects are introduced:

- **main items** (so-called concepts) like definitions, rules, axioms, theorems and proofs
- **auxiliary items** (so-called satellites) like text-items (motivations, introductions, ...), examples and exercises

For connecting these learning objects the following relations are used:

“The domain prerequisite relation expresses a mathematical dependency between concepts, ie, it describes which concepts are mathematically necessary in order to define the current one. The for relation links satellite items to concepts and definitions to symbols. For instance, an example or exercise may illustrate or train a related definition. The against relation describes an item as a counter example or misconception of the related concept. The is a relation serves to represent mathematical hierarchies and special cases, eg, a “function” is a specific “relation”.

Note that several (mathematically equivalent) definitions may exist for one symbol, eg, the convergence of a function can be defined via $\varepsilon - \delta$ formulae or via sequences. Definitions and theorems can be different for different learning contexts or fields too.” (Ullrich; Libbrecht 2008, p.7)

It is not possible yet to define e.g. the auxiliary items in the domain ontology, because we don't know the concrete learning objects of the content partners. Instead, we “only” have to identify the concepts, we will provide in Math-Bridge. Thus, the domain ontology just defines the **symbol**. They are represented by classes: The fact that class A is a subclass from B means

in ActiveMath that symbol A has an is_a-Relation to symbol B.

Accordingly, the domain ontology of Math-Bridge will consist of symbols and is_a-relations. During the project we want to keep an eye on coverage using the ontology. Therefore main items and auxiliary items must be inserted into the ontology too. The insertion can be done automatically.

3.2 The domain ontology

In this section we will describe the results of the domain-ontology that we have developed for Math-Bridge. We start with considering domain-ontologies that were already realised, e.g. in ActiveMath-projects. After that, we discuss in detail the our new domain-ontologies. Finally we will describe further work related to the domain-ontology that has still to be done.

At the current state of the project we have developed the following building blocks of the domain ontology:

- reconstruction of the implicit ontology of the fraction system in ActiveMath
- reconstruction of the implicit ontology of the calculus system in LeActiveMath
- tentative domain ontology for the bridging course material from the university of Kassel
- transformation of the mathematical taxonomy for Mathematical Sciences Education into a domain ontology

These four ontologies are provided as OWL-files in http://subversion.dfki.de/math-bridge/public/WP01_Pedagogical_Preparation/Deliverables/ontologies/.

In the next step these ontologies have to be integrated and enriched by taking into account all the learning objects of the other content partners. Also it has to be decided how fine-grained the final ontology has to be.

3.2.1 Existing implicit ontologies in the ActiveMath-context

In a series of projects, where ActiveMath were used, domain ontologies are implicitly developed and used. But these ontologies are not given explicitly. Instead, they are implicitly contained in the metadata.

In particular, there are two projects that we will build upon in Math-Bridge. Also in these two projects the ontology is not defined explicitly, but can be reconstructed from the metadata of the learning objects, especially from the metadata of the symbol and their is_a-relations.

1. First we have the content developed by E. Kessler about fractions in school mathematics (<http://www.activemath.org/Content/SchoolMathematicsFractions>). For this content the domain ontology differentiates between the three types concepts, rules and Grundvorstellungen. The onlogy was reconstructed, see the Protege-Project: http://subversion.dfki.de/math-bridge/private/WP01_Pedagogical_Preparation/Deliverables/ontologies/fraction-ontology/Bruch.pprj
2. Second we have the project LeActiveMath about differential calculus. Here the domain ontology is more complex. 19 major symbols were defined to structure the content.

A closer look shows a creative way to use the is_a-relation. For instance there is a major symbol `abs_conv_series_symbols` with the description “symbols in connection with absolutely convergent series”. In this category you’ll find the symbols for “absolutely convergent”, “cauchy product”, “conditionally convergent” and “rearrangement”.

In other words, the meaning of the is_a-relation is better described here as “in connection

with”. This wider use as compared to “is a special mathematical case of” allows a broader and more reasonable application of the `is_a`-relation. We will use this in our project.

The reconstructed ontology can be found in the following owl-file for LeActiveMath:

http://subversion.dfki.de/math-bridge/public/WP01_Pedagogical_Preparation/Deliverables/ontologies/LeAM-ontology/LeAMcalculus.owl

3.2.2 The domain-ontology for Kassel material

In this section we first explain the process of the collaborative work and then we describe its result. Based on the ideas we got from the two existing ontologies in ActiveMath, we first started developing a domain ontology for the content from Kassel. This is completed by the results of the content selection process, which we will reinterpret as domain ontology.

In the first step we scanned the material provided by Kassel with respect to the concepts considered in the material. 150 concepts could be identified. After a closer look we deleted those concepts, which aren't important enough to become a symbol. Of course, the skipped concepts can later be related to the chosen symbols. After this process the number of symbols remained at about 100.

Finally, analysing the chosen symbols for the Kassel content with regard to the original `is_a`-relation leads (cf. http://subversion.dfki.de/math-bridge/public/WP01_Pedagogical_Preparation/Deliverables/ontologies/kassel-ontology/kassel-ontology.owl) to 6 main topics or main symbols the users have to learn:

1. set (particularly the rational numbers)
2. power
3. equation
4. function
5. sequence
6. limit

The six main symbols contain a number of sub-symbols. These sub-symbols are more than examples, hence they have to get own symbols.

In the annex the main symbols and their sub-symbols are presented together with their `is_a` relations.

A closer look at those symbols shows that the algebraic transformations aren't mentioned there. We subsume these transformations as rules under the set Q . The subsumed rules are: binomial theorem, associative law, distributive law, commutative law and algorithms for fractions.

Additionally to the main symbols there are symbols without an `is_a`-relation to other symbols. These symbols cannot be categorized in a useful way. For instance this is the case for the symbols `value`, `denom_number_system`, `pole_line` or `inequality`. Furthermore we defined some symbols with sub-symbols, which don't belong to the main symbols. For instance these are proof strategies or assertions or logic operators. Also these symbols can be found in the annex.

3.2.3 Transformation of the mathematical taxonomy for Mathematical Sciences Education into a domain ontology

We discussed the ontologies described above and came to the following problems:

1. As we have mentioned above, we could identify only 6 topics in the Kassel ontology which are highly networked on their own, but not among each other. Moreover many topics stand alone and are not networked at all.
2. The categorization isn't obvious. For instance the denominational number system, prime numbers and divisibility aren't connected, although they are all topics in arithmetic. In the context of LeActiveMath this problem is solved by tricky main symbols such as described before. This does not solve the problem, it avoids it. Searching for a definition within ActiveMath a user wants to know a domain the definition belongs to. Using the solution described above he will be irritated.

Due to these problems we decided to change some aspects of the framework. We generalised the `is_a`-relation to a `belongs_to`-relation. This relation expresses a “subtopic” relation. For instance combinatorics belongs to “Discrete Mathematics” and “Statistics and Probability” but not to “Functions” or “Analysis”. It includes the previous `is_a`-relation meaning “special case of”.

Hence our domain ontology has to be defined with topics/symbols and the `belongs_to`-relation. With this new definition of the framework our problems were solved:

Now we could use the taxonomy, on which the content selection is based, also as a basis for the domain ontology and created the Math-Bridge ontology as it can be found in the following owl-file: http://subversion.dfki.de/math-bridge/public/WP01_Pedagogical_Preparation/Deliverables/ontologies/tax-ontology/taxonomy.owl

The ontology for the main symbols can be found in the Annex, too. We only selected those topics for the taxonomy that we have chosen in the process of content selection.

3.2.4 Problems and future work

Taking into account the differences between mathematics at school and at university level we have to consider the following two aspects:

1. In some cases the meaning of a concept develops during the process of learning. For instance the understanding of cosine as a mathematical concept starts in school education mostly with the meaning of a ratio of 2 sides in a right-angled triangle and usually develops into a definition of the cosine as a x-coordinate of a point on the unit circle. Later on the cosine is defined as a periodic function. These evolution has to be represented in the ontology.
2. We use the names of concepts like they are taught in school. As the example of linear function shows, the name “linear function” means another concept at school than at university. At school, “linear functions” are polynoms with degree one, in university a linear function is restrained to a polynoms with degree one without a constant member.

So our meaning associated to linear function is a polynom with degree one.

In the future, the ontology within Math-Bridge must be tested during the metadata enrichment process. In this process it may turn out that parts of the ontology are e.g. not “rich” enough: further relation and sub-symbols may have to be added.

Another aim is to integrate the existing ontologies of the school mathematics fraction project and the LeActiveMath calculus project into the Math-Bridge ontology.

4 Summary

The target competencies in Math-Bridge are characterized by the following three dimensions as described in chapter two and three:

Dimension 1: mathematical competence cluster

- Technical competency
- Mathematical problem solving competency
- Modelling competency
- Communication and reasoning competency

Dimension 2: achievement levels

- reproduction
- connection
- reflection

Dimension 3: domain ontology

Additionally 5 difficulty levels are distinguished for exercises.

The dimensions allow a complete classification of the mathematical content and of the competencies that are needed within Math-Bridge. Moreover they give the possibility for easy and understandable user feedback concerning their knowledge and competencies.

Based on this report the next steps of the project are feasible: Clearly, the content enrichment in work package 2 and the assessment selection can start. Furthermore, our presentation will be the basis for developing criteria for evaluation (T9.3 and T9.5) as well as for the gap detecting in task 3.6. Finally, some adaptations of ActiveMath are needed due to our results, e.g. an adaptation of the search (T4.5), the student model (T7.4) and the measurement of the student's competencies in T7.2.

Further project developments and evaluation results might require adjustments of the presented model based upon a re-discussion between the pedagogical partners in WP1.

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6 Annex

6.1 ActiveMath: Mappings of the proposed cognitive processes to revised Bloom and PISA

Proposed category	Proposed process	PISA	Revised Bloom
Remember	Recognize Recall	n/a n/a	Recognize Recall
Represent	Interpret Exemplify Transform Summarize	Represent/Think Represent Represent Think	Interpreting Exemplifying Interpreting Summarizing
Compare	Find commonalities Find differences Classify Infer Order	Think Think Think Think Think	Comparing Comparing Classifying Inferring Comparing
Solve	Estimate Apply algorithm Apply tool	Solve Solve Tools	Inferring Executing n/a
Analyze	Check Differentiate Organize Attribute	Solve Think/Model n/a (Think/Argue)	Checking Differentiate Organize Attribute
Model	Decode Encode Generate Produce	Model Model Model/Think n/a	Interpreting Interpreting Generating Producing
Communicate	Describe Explain Critique Prove	Communicate Communicate Argue Argue	Explaining Explaining Critiquing Inferring
Meta-cognition	Reflect Help seeking Search for information Detect errors Plan Self-monitor Self-explain	(Solve/Model) n/a n/a (Solve) (Solve) (Solve) (Argue/Communicate)	n/a n/a n/a Checking Planning Checking (Explaining)

Source: Faulhaber et al., p. 9.

6.2 detailed description of the competency levels according to the competencies in PISA

A clear progression through these levels is apparent in the way in which the individual mathematical competencies specified in the PISA mathematics framework play out as literacy levels increase.

For example, the competency thinking and reasoning is observed to follow a progression through six stages:

1. Follow direct instructions and take obvious actions.
2. Use direct reasoning and literal interpretations.
3. Make sequential decisions, interpret and reason from different information sources.
4. Employ flexible reasoning and some insight.
5. Use well developed thinking and reasoning skills.
6. Use advanced mathematical thinking and reasoning.

The communication competency follows this progression:

1. Follow explicit instructions.
2. Extract information and make literal interpretations.
3. Produce short communications supporting interpretations.
4. Construct and communicate explanations and argument.
5. Formulate and communicate interpretations and reasoning.
6. Formulate precise communications.

For modelling, the following development is observed as literacy levels increase:

1. Apply simple given models.
2. Recognise, apply and interpret basic given models.
3. Make use of different representational models.
4. Work with explicit models, and related constraints and assumptions.
5. Develop and work with complex models; reflect on modelling processes and outcomes.
6. Conceptualise and work with models of complex mathematical processes and relationships; reflect on, generalise and explain modelling outcomes.

For problem posing and problem solving, the following development is observed as literacy levels increase:

1. Handle direct and explicit problems.
2. Use direct inference.
3. Use simple problem-solving strategies.
4. Work with constraints and assumptions.
5. Select, compare and evaluate appropriate problem-solving strategies.
6. Investigate and model with complex problem situations.

In the case of the competency representation, the following development is observed as literacy levels increase:

1. Handle familiar and direct information.
2. Extract information from single representations.
3. Interpret and use different representations.
4. Select and integrate different representations and link them to real world situations.
5. Make strategic use of appropriately linked representations.
6. Link different information and representations and translate flexibly among them.

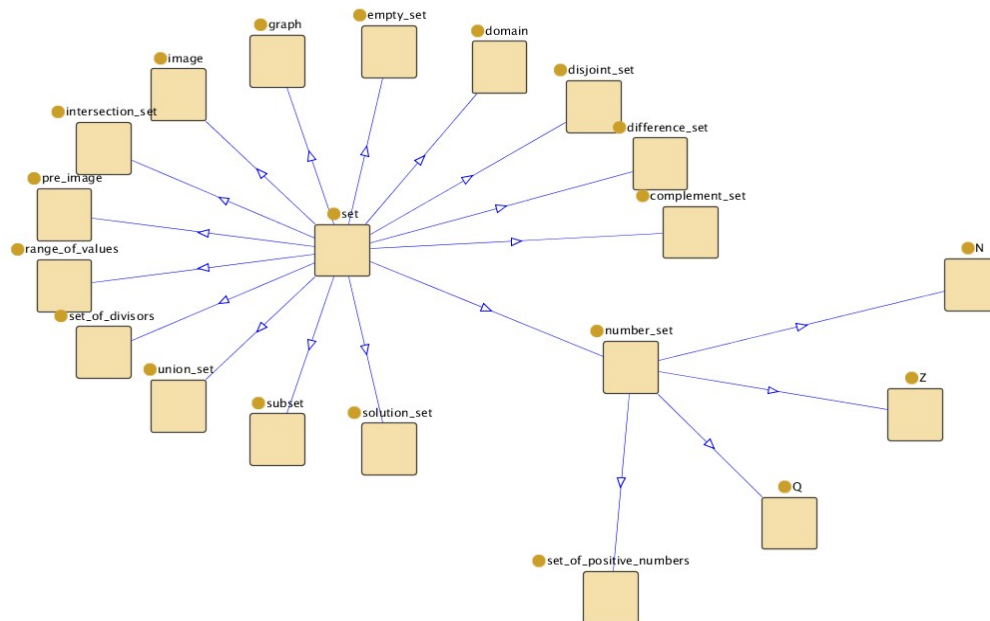
Likewise, for using symbolic, formal and technical language and operations, these stages of development are observed:

1. Apply routine procedures.
2. Employ basic algorithms, formulae, procedures and conventions.
3. Work with symbolic representations.
4. Use symbolic and formal characterisations.
5. Mastery of symbolic and formal mathematical operations and relationships.

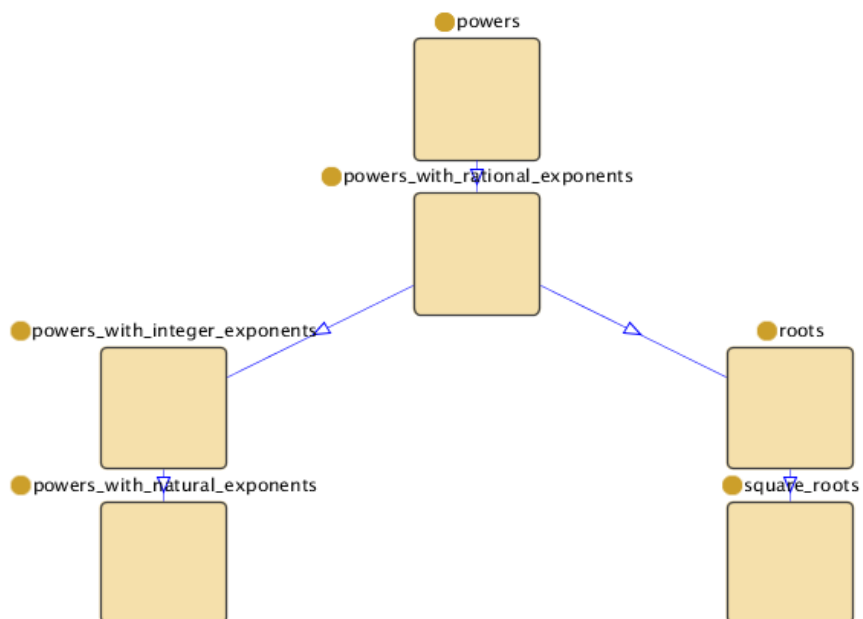
fig.: OECD 2005, pp. 262-263.

6.3 domain ontology for the Kassel material

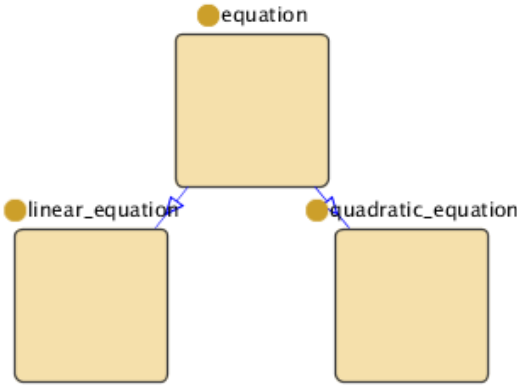
6.3.1 Set



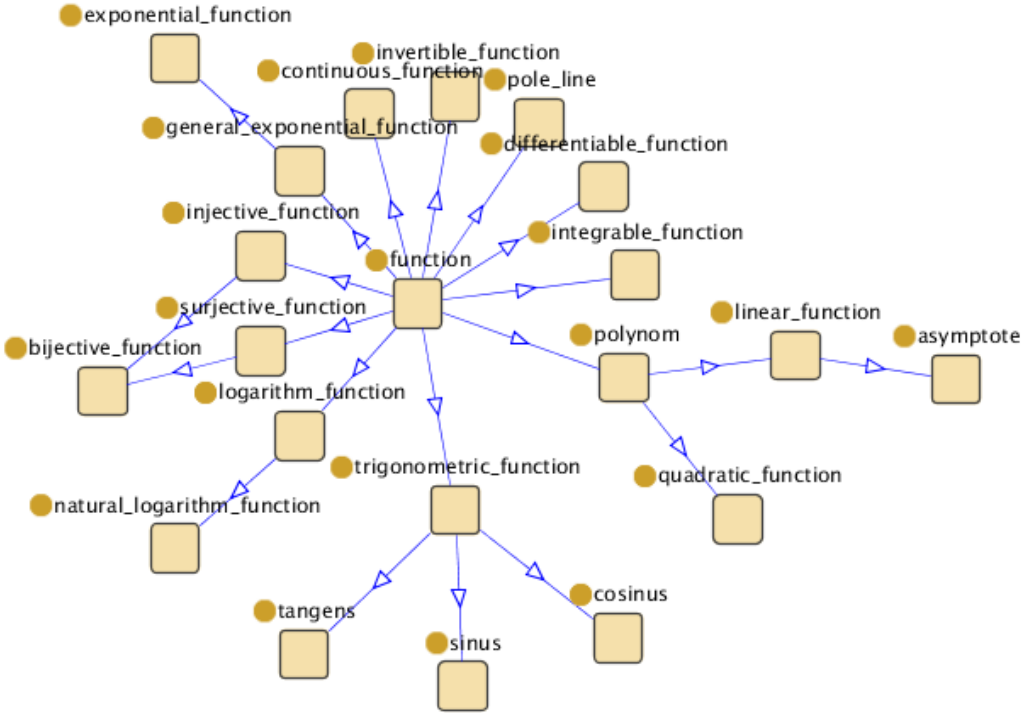
6.3.2 Power



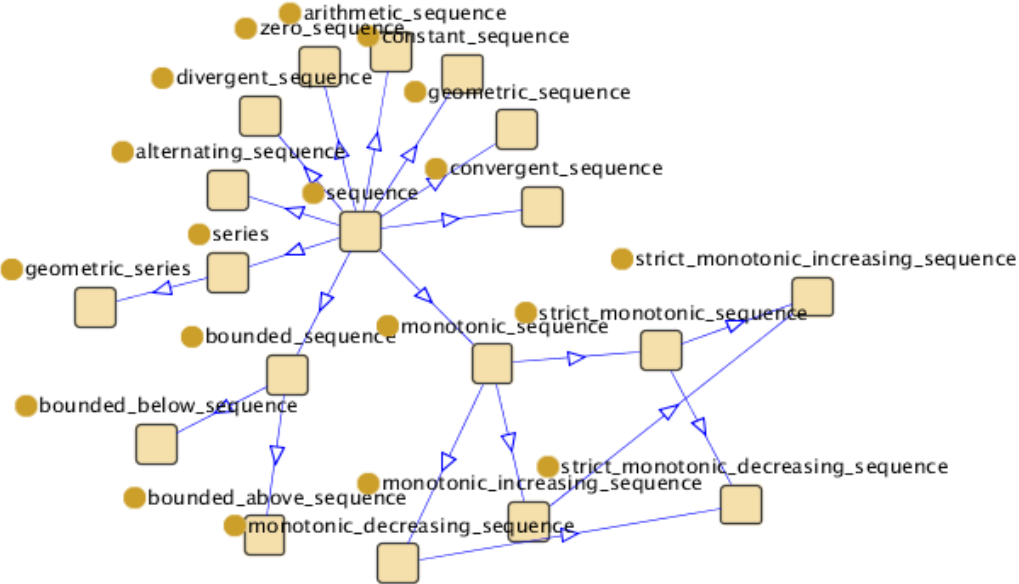
6.3.3 Equation



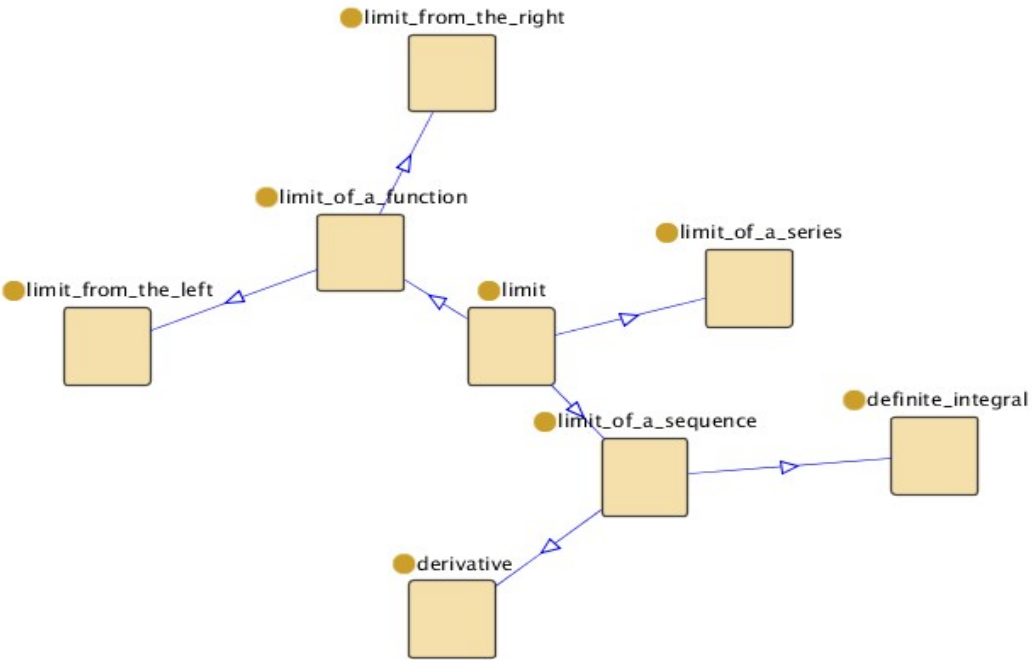
6.3.4 Function



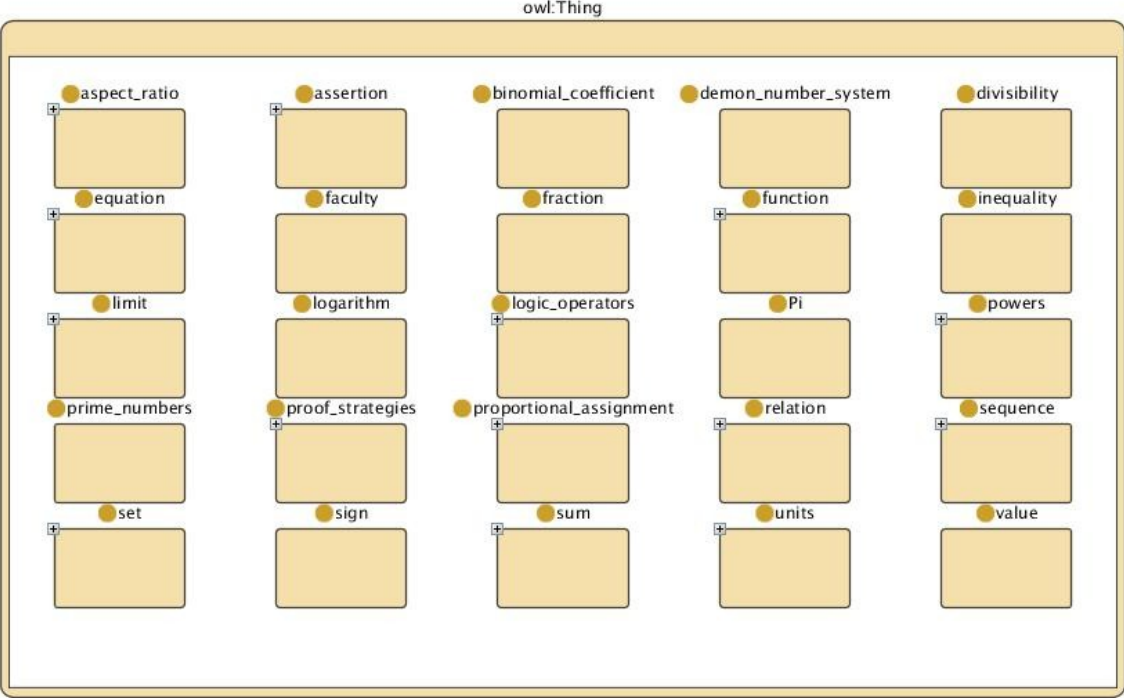
6.3.5 Sequence



6.3.6 Limit

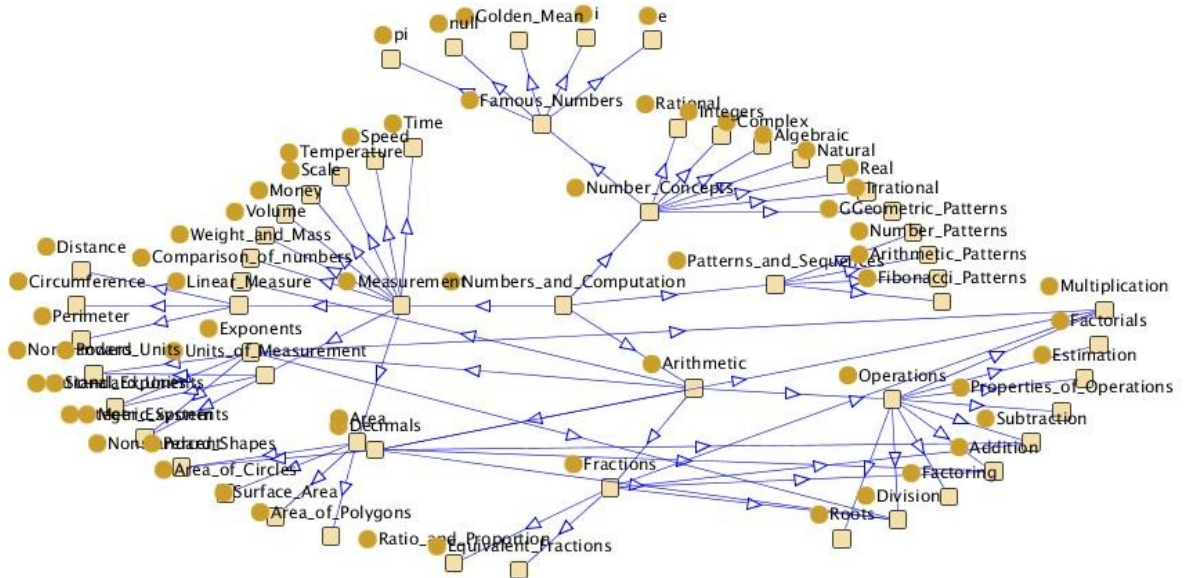


6.3.7 Overview of the symbols

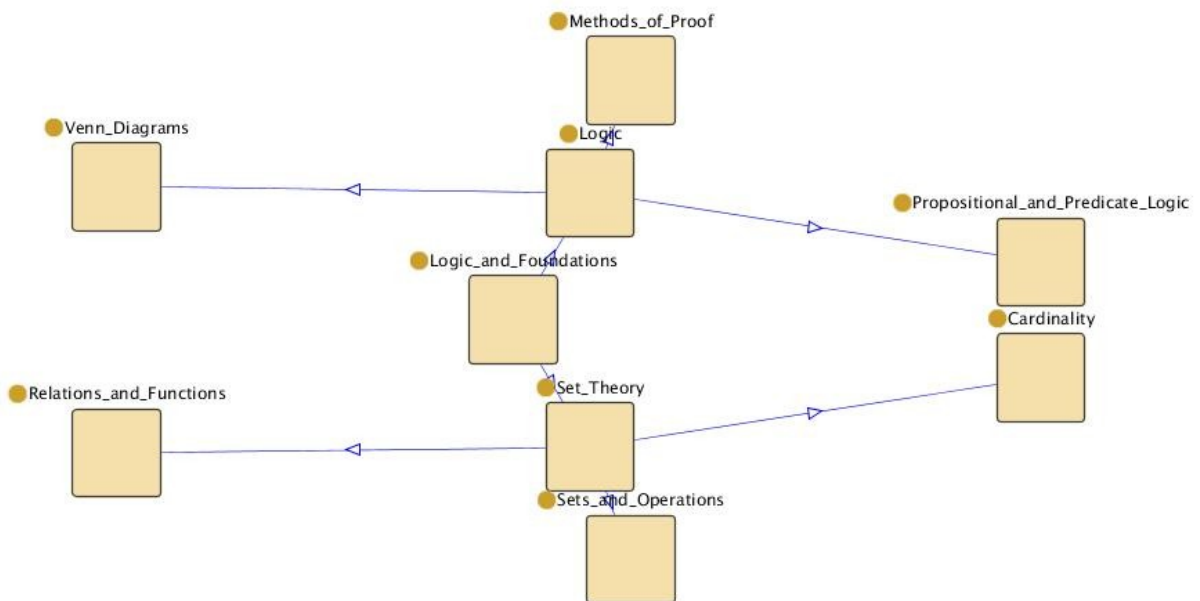


6.4 Taxonomy

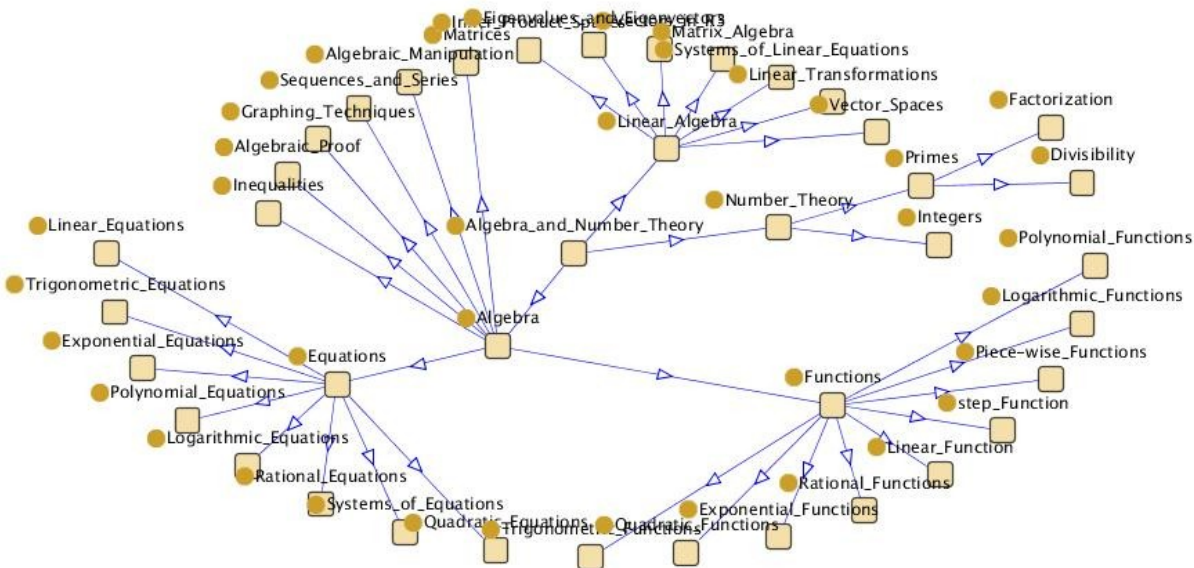
6.4.1 Numbers and Computation



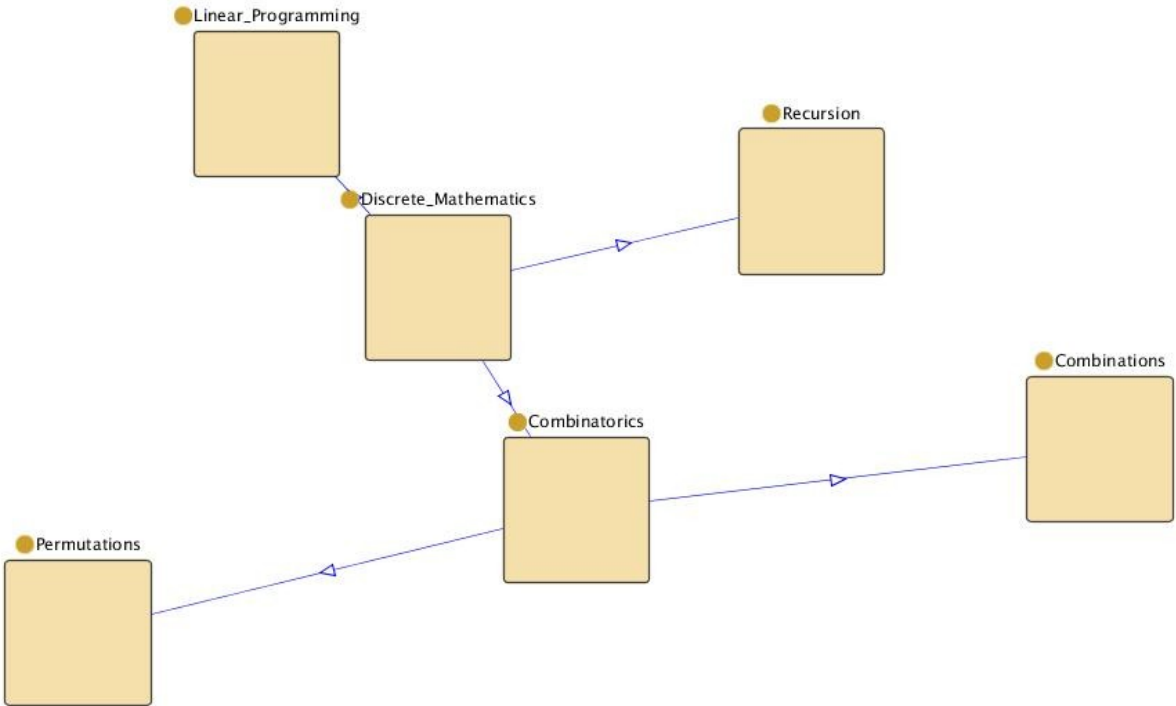
6.4.2 Logic and Foundations



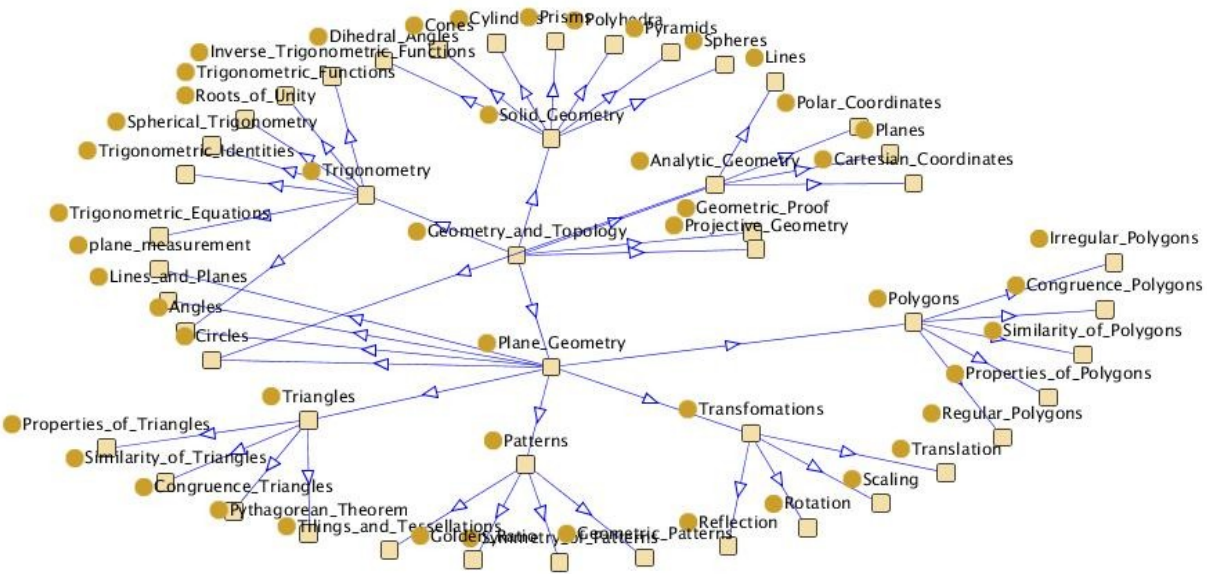
6.4.3 Algebra and Number Theory



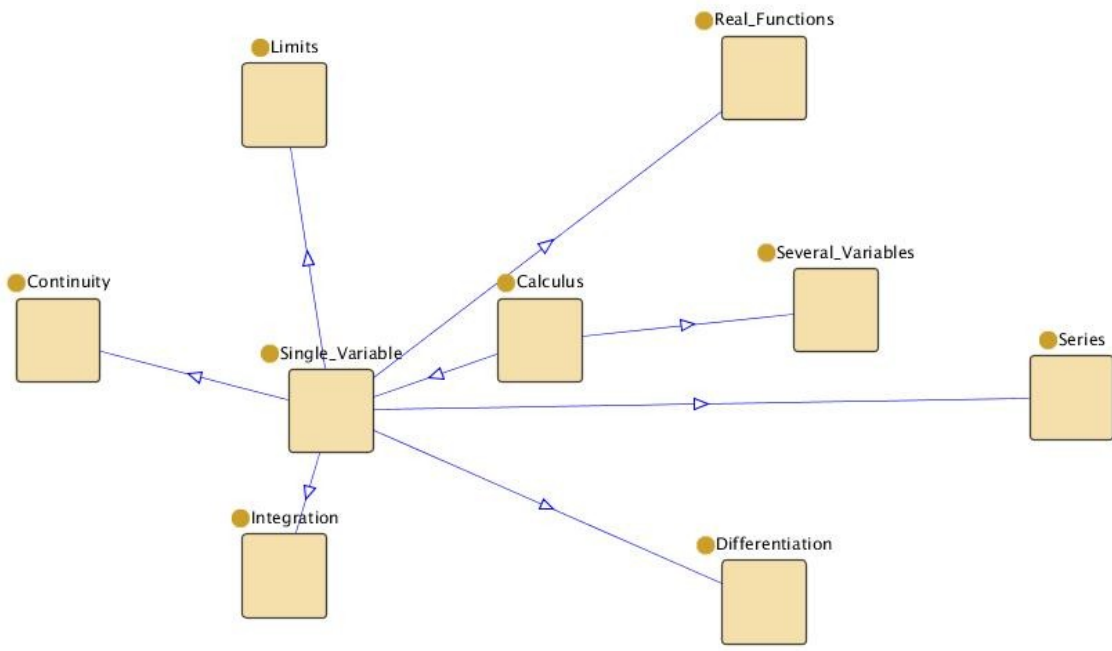
6.4.4 Discrete Mathematics



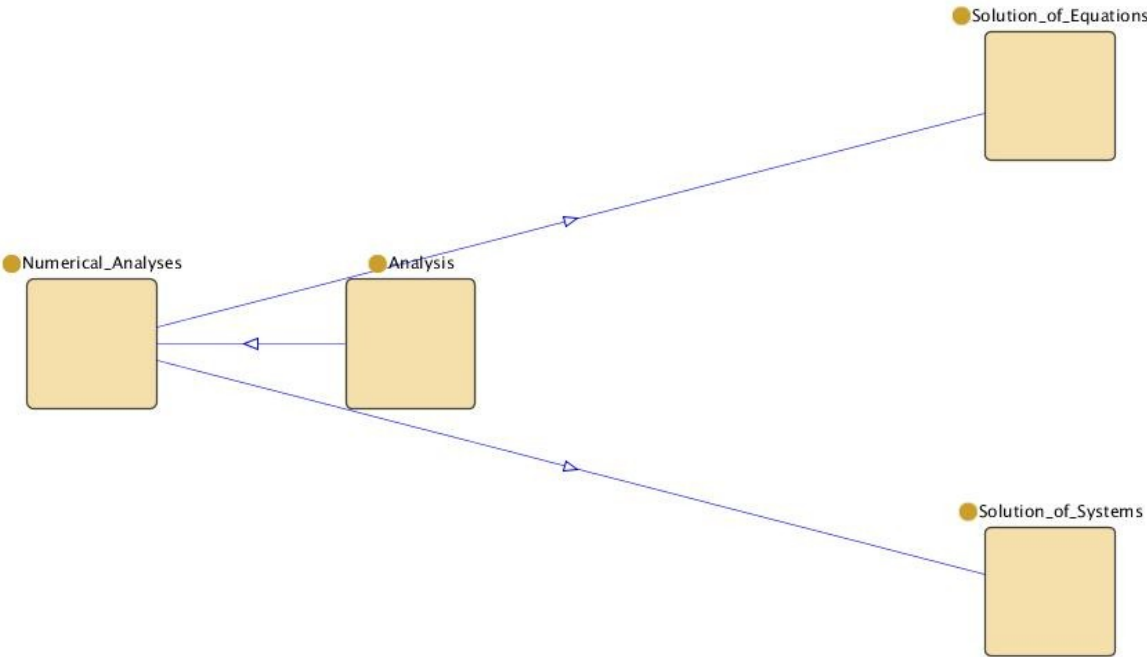
6.4.5 Geometry and Topology



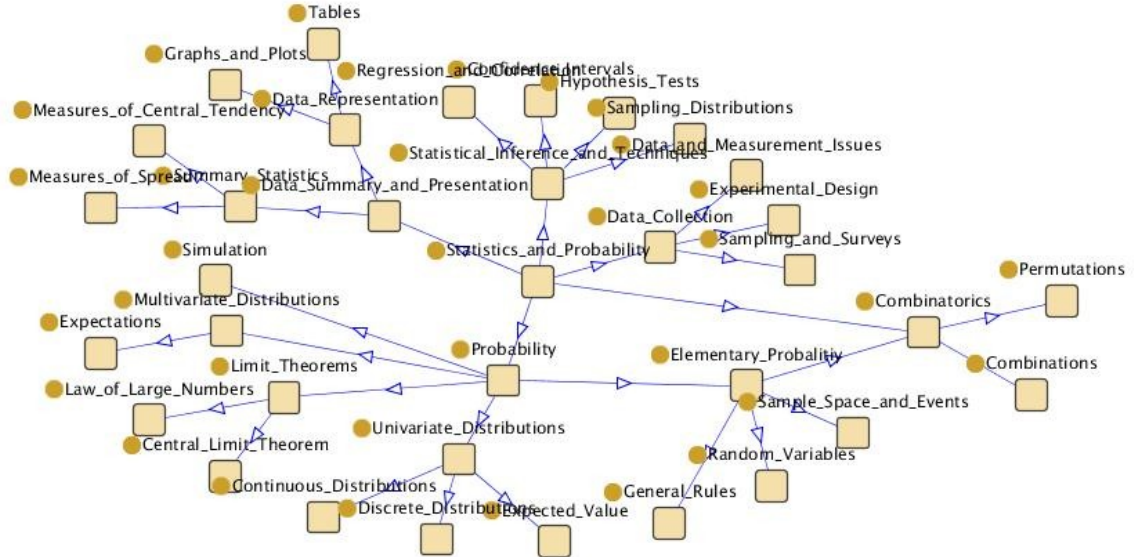
6.4.6 Calculus



6.4.7 Analysis



6.4.8 Statistics and Probability



6.4.9 Applied Mathematics

